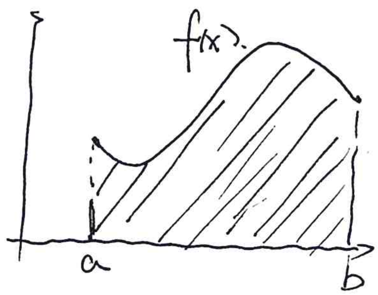
# §4.1 Area and distance.

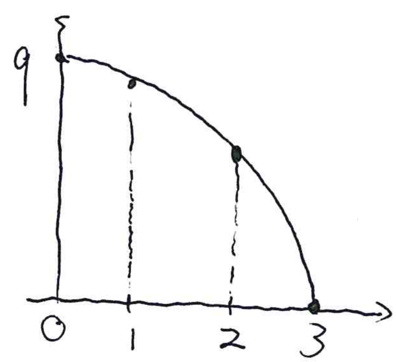
- Goal: Use equally-spaced rectangles to estimate the area.




- Left / Right endpoints sums
- Upper / Lower sums

Over / Under - estimates

- eg1. Use three rectangles of equal width to estimate the area of $y=-x^2+9$ between $x=0$ and $x=3$, above x-axis.




| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | 9 | 8 | 5 | 0 |

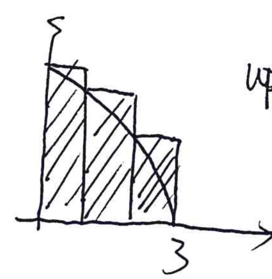
interval $[0,1]$, $[1,2]$, $[2,3]$

width. $\frac{3-0}{3}=1$

Leftendpoints sum: $1\cdot f(0)+1\cdot f(1)+1\cdot f(2)=9+8+5=22$

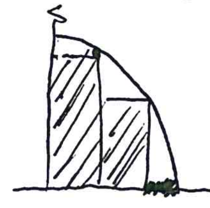
Right endpoints sum: $1\cdot f(1)+1\cdot f(2)+1\cdot f(3)=8+5+0=13$

Upper sum (Over estimate) : $1\cdot f(0)+1\cdot f(1)+1\cdot f(2)=22$




upper sum

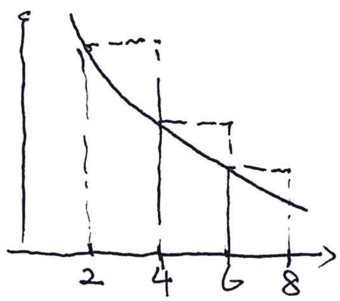
Lower sum (Under estimate) : $1\cdot f(1)+1\cdot f(2)+1\cdot f(3)=13$



lower sum

eg 2. Using three equally-spaced rectangles of equal width, find
(S17) the upper sum approximation of the area between the curve $y=\frac{1}{x}$ and the x-axis from $x=2$ to $x=8$.

s/n:




Total interval: $[2, 8]$. Total width: $8-2=6$

number of subinterval: 3. Width of each subinterval: $\frac{6}{3}=2$

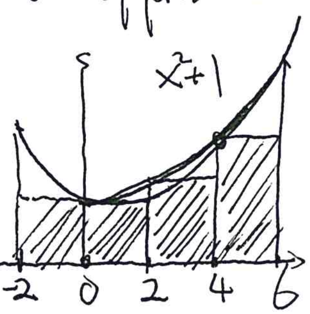
Three intervals: $[2,4]$, $[4,6]$, $[6,8]$

Upper Sum (= left endpoint in this case)

$$= 2\cdot(f(2)+f(4)+f(6)) = 2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right) = \boxed{\frac{11}{6}}$$

☆eg 3 Find the ~~tot~~ under-estimate using 4 equally-spaced rectangles to approximate the area between $y=x^2+1$, x-axis from $x=-2$ to $x=6$

s/n:




Underestimate: choose the rectangles BELOW the curve

1st interval: $[-2, 0]$. height: $f(0)$.

2nd interval: $[0, 2]$. height: $f(0)$

3rd interval: $[2, 4]$ height: $f(2)$.

4th interval: $[4, 6]$ height: $f(4)$

Lower sum: $2\cdot[f(0)+f(0)+f(2)+f(4)] = 2\cdot[1+1+5+17] = 48$

Remark: Overestimate: $2\cdot[f(-2)+f(2)+f(4)+f(6)] = 2\cdot[5+5+17+37]$

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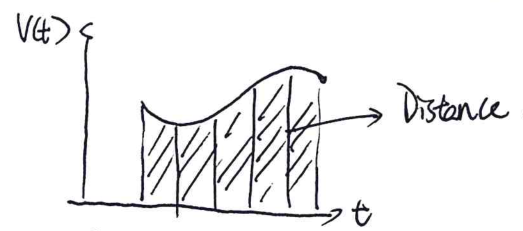
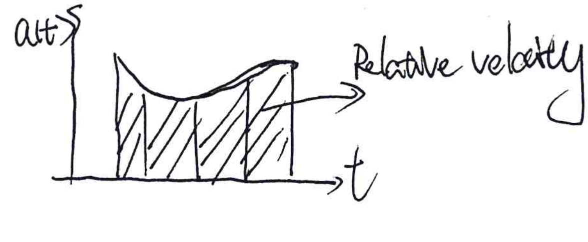
(Hints for webwork 4, 5)

- Distance = Area below the velocity $V(t)$.




- Relative velocity = Area below acceleration $a(t)$



§4.1. Part II. § Appendix E Sigma Notation

$$\overset{\text{end}\to\ n}{\underset{\text{index}\to\ i=m\ \uparrow\text{start}}{\text{sum}\to\sum}} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-2} + a_{n-1} + a_n$$

($\Sigma$ read as Sigma).

eg.1. $\sum_{i=1}^{5} 2i = 2\cdot1 + 2\cdot2 + 2\cdot3 + 2\cdot4 + 2\cdot5$

eg.2. Write $1+2+4+8$ in the sigma notation. Hint: $1=2^0$
$2=2^1$
$4=2^2$
$8=2^3$

Formula for ith term: $2^i$

i starts from 0, ends at 3.

$1+2+4+8 = \sum_{i=0}^{3} 2^i$

eg.3 Find the sum $\sum_{i=99}^{100} \frac{2}{i-98}$

sln: $\sum_{i=99}^{100} \frac{2}{i-98} = \frac{2}{99-98} + \frac{2}{100-98} = \frac{2}{1} + \frac{2}{2} = \boxed{3}$.

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Formulas: (formula sheet)

① $\sum_{i=1}^{n} c = c\cdot n$, ② $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, ③ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

eg4. $\sum_{i=1}^{20}(-3) = -3\cdot20 = -60$

$(= \underbrace{-3 -3 -3 - \cdots -3}_{\text{20 copies of } (-3)})$

eg5 Find $\sum_{i=1}^{30}(3+2i) \xlongequal{\text{linear}} \sum_{i=1}^{30} 3 + \sum_{i=1}^{30} 2i$

(Fi4)

$$= 3\cdot 30 + 2\cdot \sum_{i=1}^{30} i$$

$$= 90 + 2\cdot \frac{30\cdot(30+1)}{2}$$ (Formula ②, n=30

$$= 90 + 30\cdot 31 = 90+930 = \boxed{1020}$$

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Hints for webwork:

※5. $\cos 0 = 1$, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, ....

$\cos i\pi = (-1)^i$, $i = 0, 1, 2, \cdots$

※6. $\sum_{i=1}^{55}(-3i^2) = -3\cdot\sum_{i=1}^{55} i^2 = -3\cdot\frac{55\cdot(55+1)\cdot(2\cdot 55+1)}{6}$

(Formula)

$\sum_{i=5}^{55}(-3i^2) = \left[\sum_{i=1}^{55}(-3i^2)\right] - \left[\sum_{i=1}^{4} -3i^2\right]$

※8. $\sum_{i=1}^{n}(5i^2+7i) = \sum_{i=1}^{n} 5\cdot i^2 + \sum_{i=1}^{n} 7i = 5\cdot\sum_{i=1}^{n} i^2 + 7\sum_{i=1}^{n} i = 5\cdot\frac{n(n+1)(2n+1)}{6} + 7\cdot\frac{n(n+1)}{2}$

※14. $\sum_{i=1}^{50} i(7i+2) = \sum_{i=1}^{50}(7i^2+2i) = \sum_{i=1}^{50} 7i^2 + \sum_{i=1}^{50} 2i$ Formula ②, ③

$$= 7\cdot\frac{50\cdot(50+1)(2\cdot 50+1)}{6} + 2\cdot\frac{50(50+1)}{2}.$$

※11. Write $4+8+12+16+20+24$ as a sigma notation

$= 4\cdot 1 + 4\cdot 2 + 4\cdot 3 + 4\cdot 4 + 4\cdot 5 + 4\cdot 6 = \sum_{i=1}^{6} 4i$

$= 4(1+0) + 4(1+1) + \cdots + 4(5+1) = \sum_{i=0}^{5} 4(i+1)$

※12. $\sum_{i=0}^{4}(-1)^i = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 = 1 - 1 + 1 - 1 + 1 = \boxed{1}$

## §4.2. The Definite Integral.

• The Definite Integral of $f(x)$ from $x=a$ to $x=b$ is denoted by

$$\int_a^b f(x)\,dx := \text{"Area under the curve"} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x,\quad \Delta x=\frac{b-a}{n},\ x_i=a+i\cdot\frac{b-a}{n}.$$

Remark: $\int$ : integral notation  $a$: lower limit. $b$: upper limit. $dx$: integral w.r.t. $x$ variable

Remark: Integral "=" Area under the curve, only depends on $f$ and $a,b$ (is a number)

The variable $dx$ can be changed to any other variable. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$

• Give a Riemann Sum, how to find the corresponding Integral $\int_a^b f(x)\,dx$?

eg.1. The limit $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{7+\frac{4i}{n}}\cdot\frac{4}{n}$ is the limit of a Riemann Sum for a certain definite integral $\int_a^b f(x)\,dx$. Find the exact form of $\int_a^b f(x)\,dx$.

Solution: $\frac{4}{n}$ plays the role of $\Delta x=\frac{b-a}{n}$, i.e., $b-a=4$. the interval is 4 units long.

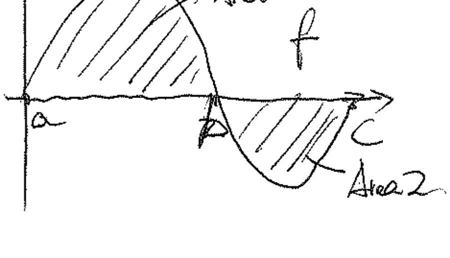
$\frac{1}{7+\frac{4i}{n}}$ plays the role of $f(x_i)$, which suggests $f$ is $\frac{1}{\ }$

Let $f(x)=\frac{1}{x}$. And our $x_1=7+\frac{4}{n}$, $x_n=7+4$, $x_i=7+i\cdot\frac{4}{n}$.

Pick $a=7$, $b=11$. Then $\int_7^{11}\frac{1}{x}\,dx$ fits the Riemann Sum.

Remark: The answer is not unique. One can check $\int_0^4 \frac{1}{7+x}\,dx$ is also correct.

• The integral represents "the area with signs". If the curve is under the x-axis, the area is considered as negative.



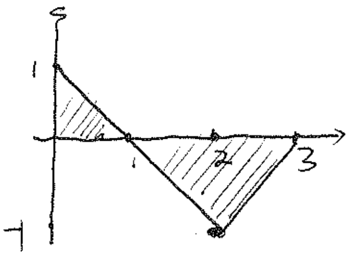

$$\int_a^b f(x)\,dx = \text{Area}1. \qquad \int_b^c f(x)\,dx = -\text{Area}2$$

$$\int_a^c f(x)\,dx = \text{Area}1 - \text{Area}2.$$

eg.2. Consider the function $f(x) = \begin{cases} 1-x & 0 \le x \le 2 \\ x-3 & 2 \le x \le 3 \end{cases}$

Use the graph of $f(x)$ on $[0,3]$ to find $\int_0^1 f(x)dx$, $\int_1^2 f(x)dx$, $\int_0^2 f(x)dx$, $\int_1^3 f(x)dx$, $\int_0^3 f(x)dx$

Solution:

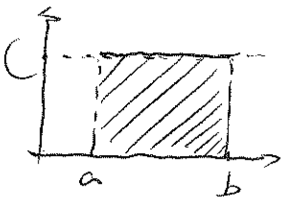



$\int_0^1 f(x)dx = \frac{1}{2}\cdot 1 \cdot 1 = \frac{1}{2}$, $\int_1^2 f(x)dx = -\frac{1}{2}$, $\int_1^3 f(x)dx = -1$

$\int_0^2 f(x)dx = \frac{1}{2} - \frac{1}{2} = 0$. $\int_0^3 f(x)dx = \frac{1}{2} - 1 = -\frac{1}{2}$.

- Some basic integrals from the graph:

Rectangle $\int_a^b C\,dx = C(b-a)$.

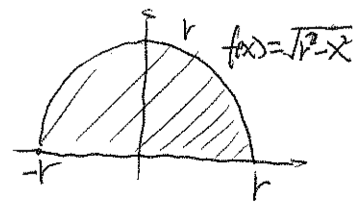


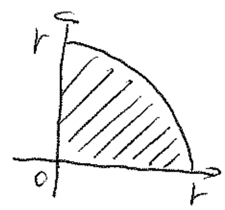

$f(x) = C$ (constant $C$)

Area $= C(b-a)$.

Half/Quarter Disk: $\int_{-r}^{r} \sqrt{r^2-x^2}\,dx = \frac{1}{2}\pi r^2$

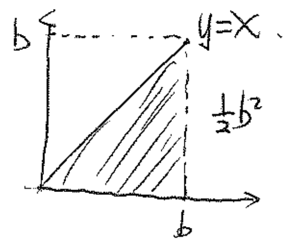
$\int_0^r \sqrt{r^2-x^2}\,dx = \frac{1}{4}\pi r^2$

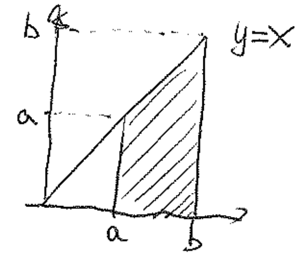



Triangle / Trapezoid: $\int_0^b x\,dx = \frac{1}{2}b^2$

$\int_a^b x\,dx = \frac{1}{2}b^2 - \frac{1}{2}a^2$

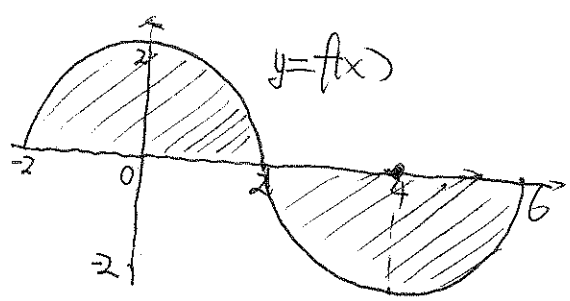



eg.3. Suppose the graph of $y=f(x)$ is as follow.

All curves are half circles with radius 2.

Find. $\int_{-2}^{2} f(x)dx$, $\int_0^2 f(x)dx$, $\int_0^4 f(x)dx$




$\int_{-2}^{2} f(x)dx = \frac{1}{2}\pi\cdot 2^2 = 2\pi$

$\int_0^2 f(x)dx = \frac{1}{4}\pi\cdot 2^2 = \pi$

$\int_0^4 f(x)dx = \frac{1}{4}\pi\cdot 2^2 - \frac{1}{4}\pi\cdot 2^2 = 0$

$\int_2^6 f(x)dx = -\frac{1}{2}\pi\cdot 2^2 = -2\pi$

$\int_0^6 f(x)dx = \pi - (2\pi) = -\pi$. $\int_{-2}^{6} f(x)dx = 2\pi - 2\pi = 0$

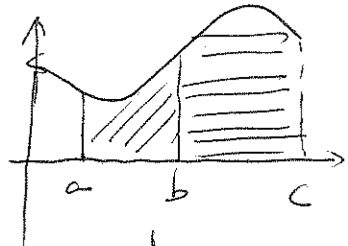
• Properties of Definite integrals

① $\int_a^a f(x)dx = 0$. (The area is zero if the upper and lower limits coincide).

② $\int_a^b f(x)dx = -\int_b^a f(x)dx$. (Flip the lower and upper limits by adding a negative sign)

eg. $\int_5^2 2x\,dx = -\int_2^5 2x\,dx$.

☆ ③ Splitting $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.




④ linear properties $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ ; $\int_a^b C f(x)dx = C \int_a^b f(x)dx$.

☆eg4 (#16) Suppose $\int_2^5 f(x)dx = 3$, $\int_2^3 f(x)dx = -4$. Find $\int_5^3 2f(x)dx$.

Solution: $\int_5^3 2f(x)dx = \int_5^2 2f(x)dx + \int_2^3 2f(x)dx$ splitting $\int_5^3 = \int_5^2 + \int_2^3$

$= -\int_2^5 2f(x)dx + \int_2^3 2f(x)dx$. flipping $\int_5^2 = -\int_2^5$

$= -2\cdot\int_2^5 f(x)dx + 2\cdot\int_2^3 f(x)dx$. constant multiple

$= -2\cdot 3 + 2\cdot(-4) = \boxed{-14}$.

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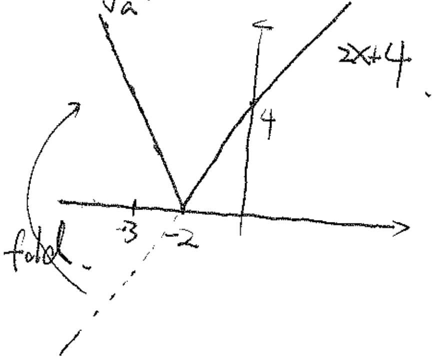
Hints for webwork:

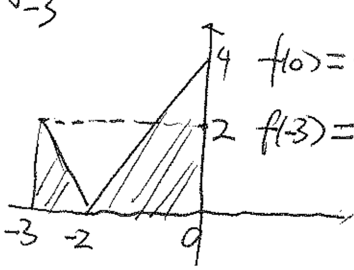
※10: ⑤ Bounds: If $A \le f(x) \le B$, then $A(b-a) \le \int_a^b f(x)dx \le B(b-a)$

☆ ※5. Graph of abstract value:

$y = |2x+4|$



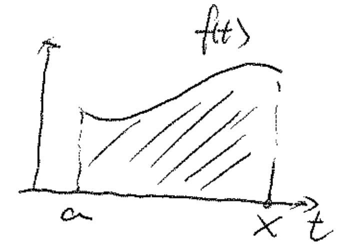

$\int_{-3}^0 |2x+4|\,dx = \frac{1}{2}\cdot 1\cdot 2 + \frac{1}{2}\cdot 2\cdot 4 = 5$




$f(0) = 4$

$f(-3) = |2\cdot(-3)+4| = 2$

# § Fundamental Theorem of Calculus.




Key formula: ① FTC P1: If $F(x)=\int_a^x f(t)\,dt$, then $F'(x)=f(x)$.

★ If $F(x)=\int_{V(x)}^{U(x)} f(t)\,dt$, then

$$F'(x)=f(U(x))\cdot U'(x)-f(V(x))\cdot V'(x). \quad \text{(chain rule form)}$$

$$\left(\int_a^{U(x)} f(t)\,dt\right)'=f(U(x))\cdot U'(x), \quad \left(\int_{V(x)}^{b} f(t)\,dt\right)'=-f(V(x))\cdot V'(x).$$

② FTC P2: If $F'(x)=f(x)$ ($F$ is one anti-D of $f(x)$)

then $\int_a^b f(x)\,dx=F(x)\Big|_a^b=F(b)-F(a)$.

③ Anti D (Integral Table):

$(n\neq -1)$: $\int_a^b x^n\,dx=\frac{1}{n+1}x^{n+1}\Big|_a^b$, $\int_a^b \cos x\,dx=\sin x\Big|_a^b$, $\int_a^b \sin x\,dx=-\cos x\Big|_a^b$

$\int_a^b \sec^2 x\,dx=\tan x\Big|_a^b$, $\int_a^b \sec x\cdot\tan x\,dx=\sec x\Big|_a^b$.

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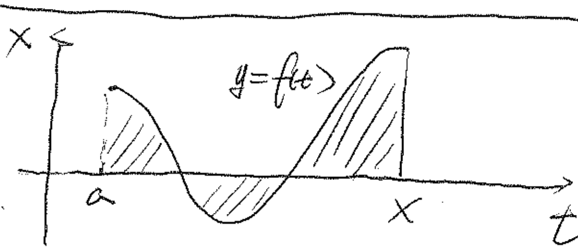
Consider $y=f(t)$, from $t=a$ to $t=x$.

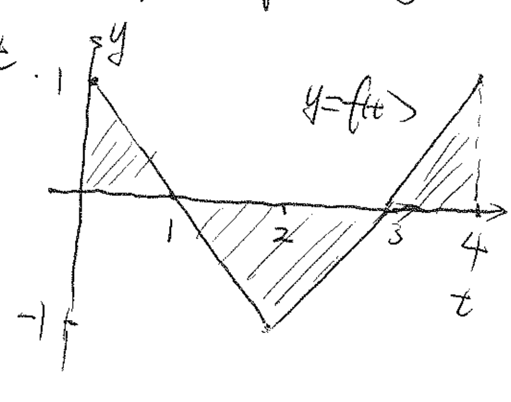



$\int_a^x f(t)\,dt$ is the area of the shadow region.

It changes as $x$ moves, therefore, is a function of the upper limit $x$.

We denote it by $F(x)=\int_a^x f(t)\,dt$. FTC P1, P2 tell us the relation between $f$ and $F$ and how to use anti D to compute a definite integral.

e.g. 1. $y=f(t)$ the graph is given below. let $g(x)=\int_0^x f(t)\,dt$.




$g(0)=\int_0^0 f(t)\,dt=0$, $g(1)=\int_0^1 f(t)\,dt=\frac{1}{2}$, $g(2)=0$

$g(3)=-\frac{1}{2}$, $g(4)=0$

★ $g(x)$ is increasing on $[0,1]\cup[3,4]$, decreasing on $[1,3]$.

• Derivative formulas:

e.g. 2. Find $\frac{d}{dx}\int_{-9}^{x}(\cos t^2 + t)\,dt = \left(\int_{-9}^{x}\cos t^2 + t\ dt\right)'$

s/n: Apply FTC P1 with $f(t) = \cos t^2 + t$. (Replace $t$ in $f(t)$ by $x$).

$\left(\int_{-9}^{x}\cos t^2 + t\ dt\right)' = \cos x^2 + x$.

e.g. 3. Let $h(x) = \int_{\tan x}^{3}\sqrt{2+t^2}\,dt$. Find $h'(x)$.

s/n: Apply FTC P1 with $u(x) = 3$ (const), $V(x) = \tan x$. $f(t) = \sqrt{2+t^2}$

$h'(x) = 0 - \sqrt{2+(\tan x)^2}\cdot(\tan x)' = -\sqrt{2+(\tan x)^2}\cdot\sec^2 x$.

↑ replace $t$ by lower limit $\tan x$.

• Evaluate the definite integral by finding the anti-D

e.g. 4. Evaluate $\int_0^{\frac{\pi}{3}} 4\cdot\sec x\cdot\tan x\,dx$.

Solution: $\int_0^{\frac{\pi}{3}} \underline{4\cdot\sec x\cdot\tan x}\,dx$. Step 1: $f(x) = 4\sec x\tan x$. Find anti-D $F(x)$ of $f(x)$

$F(x) = 4\cdot\sec x$. (since $(\sec x)' = \sec x\cdot\tan x$)

Step 2: FTC P2: $\int_0^{\frac{\pi}{3}} 4\sec x\cdot\tan x\,dx = 4\sec x\Big|_0^{\frac{\pi}{3}} = 4\sec\frac{\pi}{3} - 4\sec 0$ . Hint: $\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = 2$

$= 4\cdot 2 - 4\cdot 1 = \boxed{4}$ $\sec 0 = \frac{1}{\cos 0} = 1$

e.g. 5. Evaluate $\int_1^{\sqrt{7}} \frac{13\cdot s^4 + 5\cdot\sqrt{s}}{s^4}\,ds$

Solution: $f(s) = \frac{13\cdot s^4 + 5\cdot\sqrt{s}}{s^4} = \frac{13\cdot s^4}{s^4} + \frac{5\cdot s^{\frac{1}{2}}}{s^4} = 13 + 5\cdot s^{\frac{1}{2}-4} = 13 + 5\cdot s^{-\frac{7}{2}}$

anti-D: $F(s) = 13\cdot s + 5\cdot\frac{1}{-\frac{7}{2}+1}\cdot s^{-\frac{7}{2}+1} = 13s + 5\cdot\frac{1}{-\frac{5}{2}}\cdot s^{-\frac{5}{2}}$

$= 13s - 2\cdot s^{-\frac{5}{2}}$

FTC P2 $\Rightarrow \int_1^{\sqrt{7}} \frac{13s^4 + 5\sqrt{s}}{s^4}\,ds = (13\cdot s - 2\cdot s^{-\frac{5}{2}})\Big|_1^{\sqrt{7}} = \left(13\cdot\sqrt{7} - 2(\sqrt{7})^{-\frac{5}{2}}\right) - \left(13\cdot 1 - 2\cdot 1^{-\frac{5}{2}}\right)$

$= \boxed{13\cdot\sqrt{7} - 2\cdot(\sqrt{7})^{-\frac{5}{2}} - 11}$

- $\int_a^a f(x)dx =$ ______________; (flipping) : $\int_b^a f(x)dx =$ ______________________

- splitting : $\int_a^b f(x)dx + \int_b^c f(x)dx =$ ______________________

- $\int_a^b f(x) \pm g(x)dx =$ ____________ $\pm$ ____________, $\int_a^b C \cdot f(x)dx =$ ______________________

- $\int_a^b 1dx =$ ______________________, $\int_a^b Cdx =$ ______________________

- If $F(x) = \int_a^x f(t)\ dt$, then $F'(x) =$ .

- If $F(x) = \int_a^{u(x)} f(t)\ dt$, then $F'(x) =$ ______________________.

- If $F(x) = \int_{v(x)}^b f(t)\ dt$, then $F'(x) =$ ______________________.

- If $F(x)$ is an anti-D of $f(x)$, i.e., $F'(x) = f(x)$, then $\int_a^b f(x)\ dx =$ ______________________

- ***Antiderivative (Integral) Table:***

| $f(x)$ | $x^n, n \neq -1$ | $\cos x$ | $\sin x$ | $\sec^2 x$ | $\sec x \cdot \tan x$ |
|---|---|---|---|---|---|
| Anti-D F(x) | | | | | |
| Definite Integral $\int_a^b f(x)dx$ | | | | | |

- Graph of $y = \frac{1}{x}$

- Graph of $y = |-2x+6|$

- Graph of $y = x^2 - 2, y = -x^2 + 1$